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The Gel'fand basis and matrix elements of the graded unitary group U(m/n)

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Abstract. Projection operators with correct normalisation are constructed for obtaining the Gel'fand basis $|(\binom{[\nu]}{(m)})\rangle$ of the graded unitary group U(m/n). Each U(m/n) Gel'fand basis vector $|(\binom{[\nu]}{(m)})\rangle$ for an f-particle system uniquely corresponds to a non-standard basis vector $|[\nu](m)\rangle$ of the permutation group S(f). The matrix element of the generator $E^{i-1}{}_i$ of U(m/n) between the two Gel'fand basis vectors $|(\binom{[\nu]}{(m')})\rangle$ and $|(\binom{[\nu]}{(m)})\rangle$ is proportional to the overlap between the two non-standard basis vectors $|[\nu](m')\rangle$ and $|[\nu](m)\rangle$ of S(f). Explicit formulae are given for the normalisation constant in the projection operator as well as for the matrix elements of the generator $E^{i-1}{}_i$ of the graded unitary group U(m/n), which is the extension of the Gel'fand–Tsetlin formula for the ordinary unitary group U(m).

1. Introduction

It is known that the standard projection operator of the permutation group S(f) (or the normal unit of Rutherford (1948)) can be used to generate the Gel'fand basis of the U(m) group up to phase and normalisation from the f-particle product states (Lezuo 1972, Patterson and Harter 1976a, b, Chen *et al* 1977b). Alternatively, a non-standard projection operator of S(f) is used by Sarma and Saharasbudhe (1980) to construct the Gel'fand basis of U(m), and the Gel'fand–Tsetlin formula for the generator $E^{i-1}{}_i$ of U(m) (Gel'fand and Tsetlin 1950, Baird and Biedenharn 1963, Nagel and Moshinsky 1965) is rederived in a simple way by means of the decomposing technique for the non-standard basis of S(f) developed by them.

Recently, we introduced the Gel'fand basis for the graded unitary group U(m/n) with m(n) bosonic (fermionic) single-particle (SP) states, which is the irreducible basis (IRB) adapted to the group chain U(m/n) ⊃ U($m/n-1$) ⊃ ... U(m) ⊃ U($m-1$) ⊃ ... ⊃ U(1), and showed that it can be constructed with the help of the standard projection operator of the graded permutation group (Chen *et al* 1983a, b). A non-analytic expression was given for the normalisation constant in the standard projection operator. In this paper it will be replaced by an analytic expression.

The irreducible representation (irrep) of the group SU(m/n) has been discussed by Balentekin and Bars (1981a, b) in the context of characters, and by Han *et al* (1981) in the context of the reduced matrix elements of the SU(m/n) generators. However, an explicit expression for the irreps of the SU(m/n) Lie algebra is not available.

In this paper, we first show that each U(m/n) Gel'fand basis vector $|\binom{[\nu]}{(m)}\rangle$ for an f-particle system uniquely corresponds to a non-standard basis vector $|[\nu](m)\rangle$ of the permutation group S(f). Then we extend the Sarma–Saharasbudhe technique to decompose the non-standard basis $|[\nu](m)\rangle$. Furthermore, we show that the Gel'fand matrix element of the U(m/n) generator is expressible in terms of the overlap integral of the non-standard basis vectors of S(f), i.e.

$$\left\langle\binom{[\nu]}{(m')}\middle|E^{i-1}{}_i\middle|\binom{[\nu]}{(m)}\right\rangle=[(f_{i-1}+1)f_i]^{1/2}\langle[\nu](m')|[\nu](m)\rangle$$

where f_{i-1} and f_i are the numbers of the SP states $i-1$ and i in the basis vector $|\binom{[\nu]}{(m)}\rangle$, respectively. The overlap can be calculated by using the extended Sarma–Saharasbudhe decomposing technique for the non-standard basis of S(f). Explicit expressions are given for the matrix elements of the generator $E^{k-1}{}_k$ (boson–boson), which is nothing other than the well known Gel'fand–Tsetlin formula, the generators $E^m{}_{m+1}$ (fermion–boson) and $E^{m+k-1}{}_{m+k}$ (fermion–fermion), which are the generalisation of the Gel'fand–Tsetlin formula to the case involving both bosons and fermions. In § 5, we derive an explicit expression for the normalisation constant which is needed to construct explicitly the orthonormal Gel'fand basis of $U(m/n)$ by using the standard projection operator of S(f). In this way, the matter of constructing the Gel'fand basis and matrix elements of the graded unitary group U(m/n) is totally settled.

2. The non-standard projection operator of S(f)

2.1. *A brief review and terminology*

We begin with a brief review of the results obtained in Chen *et al* (1983a, b).

The graded coordinate permutation $(1f)^\circ$ is defined as

$$(1f)^\circ)|\varphi^1_{A_1}\varphi^2_{A_2}\ldots\varphi^f_{A_f}\rangle=\begin{bmatrix} & A_2\\ A_1 & \vdots\\ & A_f\end{bmatrix}\begin{bmatrix} & A_2\\ A_f & \vdots\\ & A_{f-1}\end{bmatrix}|\varphi^1_{A_f}\varphi^2_{A_2}\ldots\varphi^f_{A_1}\rangle, \tag{2.1a}$$

where the two square brackets are sign factors (Dondi and Jarvis 1981), whereas

$$\varphi_A=\binom{\chi_a}{\psi_\alpha}, \tag{2.1b}$$

$\varphi_1\ldots\varphi_m$ or $\chi_a,\chi_b\ldots$ representing bosonic states, while $\varphi_{m+1}\ldots\varphi_{m+n}$ or $\psi_\alpha,\psi_\beta\ldots$ represent fermionic states, and

$$\chi^i_a\chi^j_b=\chi^j_b\chi^i_a,\qquad \psi^i_\alpha\psi^j_\beta=-\psi^j_\beta\psi^i_\alpha. \tag{2.1c}$$

The $(f-1)$ permutations $(i-1,i)^\circ$, $i=2,3,\ldots f$, generate the graded coordinate permutation group $\mathring{S}(f)$. The Yamanouchi basis of $\mathring{S}(f)$ can be labelled by the graded Young tableau $\mathring{Y}^{[\nu]}_r$. The relation of $\mathring{Y}^{[\nu]}_r$ to the graded permutation $\mathring{p}$ is exactly the same as the ordinary Yamanouchi basis $Y^{[\nu]}_r$ to the ordinary permutation p.

A Gel'fand basis vector of U(m/n) is labelled by a graded Weyl tableau $\mathring{W}^{[\nu]}_{(m)}$ in which no two identical SP boson (fermion) states are permitted to occupy the same row (column).

Henceforth we will drop the attributive word 'graded' altogether for simplicity. Therefore, whenever one comes across the words permutation, permutation group, symmetric (antisymmetric), Young or Weyl tableau, one has to bear in mind that they refer to the graded permutation, graded permutation group, graded symmetric (antisymmetric), graded Young or Weyl tableau, respectively. For example, when we say the two-fermion state $\psi^1_\alpha\psi^2_\alpha$ is antisymmetric we mean that it is graded antisymmetric: $\mathring{p}\psi^1_\alpha\psi^2_\alpha = -\psi^1_\alpha\psi^2_\alpha$. Also the notations $\mathring{p}$, $\mathring{P}^{[\nu]s}_r$, $\mathring{Y}^{[\nu]}_r$ and $\mathring{W}^{[\nu]}_{[m]}$ in Chen *et al* (1983a, b) for the graded case are simplified to p, $P^{[\nu]s}_r$, $Y^{[\nu]}_r$ and $W^{[\nu]}_{(m)}$. Evidently in the boson sector of the graded space, the graded 'X' is identical to the ordinary 'X', 'X' standing for the permutation, Young tableau, ... etc.

2.2. Projection operator for the U(m/n) Gel'fand basis

It was shown (Chen *et al* 1983b) that the Yamanouchi basis of S(f) and the Gel'fand basis of U(m/n), $|Y^{[\nu]}_r, W^{[\nu]}_{(m)}\rangle$, can be obtained by applying the standard projection operator $P^{[\nu]s}_r$ of S(f) to the f-particle product states. Under the convention in § 2.1, equation (36) in Chen *et al* (1983b) can be written as

$$P^{[\nu]s}_r|f_1 \dots f_m f_{m+1} \dots f_{m+n}\rangle = R^{(\nu)s}(\{f_i\})|Y^{[\nu]}_r, W^{[\nu]}_{(m)}\rangle, \tag{2.2}$$

where $|f_1 \dots f_{m+n}\rangle$ is the f-particle product state,

$$|f_1 \dots f_m f_{m+1} \dots f_{m+n}\rangle = |\chi^1_1 \dots \chi^{f_1}_1 \dots \chi^{F_{m-1}+1}_m \dots \chi^{F_m}_m \psi^{F_m+1}_{m+1} \dots \psi^f_{m+n}\rangle, \tag{2.3}$$

$$F_i = \sum_{k=1}^{i} f_k, \qquad F_1 = f_1, \qquad F_{m+n} = f. \tag{2.4}$$

The standard projection operator of S(f) is given by

$$P^{[\nu]s}_r = \left(\frac{h_\nu}{f!}\right)^{1/2} \sum_p \langle[\nu]r|p|[\nu]s\rangle p, \tag{2.5}$$

where h_ν is the dimension of the irrep $[\nu]$ of S(f), $|[\nu]r\rangle$ and $|[\nu]s\rangle$ are the standard (i.e. Yamanouchi) basis of S(f). $R^{(\nu)s}(\{f_i\})$ is a normalisation constant (norm) depending on $\{f_i\}$,

$$R^{(\nu)s}(\{f_i\}) \equiv R^{(\nu)s}(f_1 \dots f_m; f_{m+1} \dots f_{m+n}). \tag{2.6}$$

It is convenient to use a Gel'fand symbol

$$\begin{pmatrix} [\nu] \\ (m) \end{pmatrix} = \begin{pmatrix} [\nu] \\ [\nu_{N-1}] \\ \vdots \\ [\nu_1] \end{pmatrix} \tag{2.7a}$$

to label the Gel'fand basis of U(m/n), where

$$N = m+n, \qquad [\nu] = [m_{iN}] \equiv [m_{1N}\, m_{2N} \dots], \qquad [\nu_k] = [m_{ik}] \equiv [m_{1k}\, m_{2k} \dots]. \tag{2.7b}$$

The relation between the Gel'fand symbol and the Weyl tableau $W^{[\nu]}_{(m)}$ is as follows: the Young diagram $[\nu_k]$ in (2.7a) results from deleting all the boxes containing the state labels $k+1, k+2, \dots, N$ in the Weyl tableau $W^{[\nu]}_{(m)}$.

The IRB of S(f) and U(m/n) now can be designated by

$$\left|\begin{matrix} [\nu] \\ r, (m) \end{matrix}\right\rangle \equiv |Y^{[\nu]}_r, W^{[\nu]}_{(m)}\rangle, \tag{2.8a}$$

and (2.2) becomes

$$P_r^{[\nu]s}|f_1 \ldots f_{m+n}\rangle = R^{(\nu)s}(\{f_i\}) \left| \begin{matrix} [\nu] \\ r, (m) \end{matrix} \right\rangle. \tag{2.8b}$$

The basis vector of (2.8a) satisfies the eigenequations (see equation (35) in Chen *et al* (1983b))

$$\begin{pmatrix} C(f) \\ C(s) \\ \mathscr{C}(s') \end{pmatrix} \left| \begin{matrix} [\nu] \\ r, (m) \end{matrix} \right\rangle = \begin{pmatrix} \nu \\ r \\ (m) \end{pmatrix} \left| \begin{matrix} [\nu] \\ r, (m) \end{matrix} \right\rangle, \tag{2.9a}$$

where $(C(f), C(s))$ is the CSCO-II of $\mathrm{S}(f)$, and

$$\mathscr{C}(s') = (\mathscr{C}(F_{N-1}), \ldots, \mathscr{C}(F_2), \mathscr{C}(F_1)), \tag{2.9b}$$

$\mathscr{C}(F_i)$ being the CSCO-I of the state permutation group $\mathscr{S}(F_i)$. There is a one-to-one correspondence between the eigenvalues of $C(f), C(s), \mathscr{C}(s')$ on one hand, and ν, $r, (m)$ on the other hand (Chen and Gao 1982). Hence in (2.9a) we just use ν, r, and (m) to represent the eigenvalues of $C(f)$, $C(s)$ and $\mathscr{C}(s')$ for simplicity in notation.

In the following, we need to use intrinsic permutation group $\bar{\mathrm{S}}(f)$ (Chen and Gao 1982) whose elements $\bar{p}_a$ are defined by

$$\bar{p}_a p_b = p_b p_a \qquad \text{for any } p_b \in \mathrm{S}(f). \tag{2.10}$$

The relation between the state permutation $\mathscr{P}$ and the intrinsic permutation $\bar{p}$ when acting on the product states is

$$\bar{p} = \mathscr{P}^{-1}. \tag{2.11}$$

Therefore (2.9a) can be replaced by

$$\begin{pmatrix} C(f) \\ C(s) \\ \bar{C}(s') \end{pmatrix} \left| \begin{matrix} [\nu] \\ r, (m) \end{matrix} \right\rangle = \begin{pmatrix} \nu \\ r \\ (m) \end{pmatrix} \left| \begin{matrix} [\nu] \\ r, (m) \end{matrix} \right\rangle, \tag{2.12a}$$

$$\bar{C}(s') = (\bar{C}(F_{N-1}), \ldots, \bar{C}(F_2), \bar{C}(F_1)), \tag{2.12b}$$

where $\bar{C}(F_i)$ is the CSCO-I of the subgroup $\bar{\mathrm{S}}(F_i)$ of the intrinsic permutation group $\bar{\mathrm{S}}(f)$.

Now let us introduce a non-standard projection operator of $\mathrm{S}(f)$,

$$P_r^{[\nu](m)} = \left(\frac{h_\nu}{f!}\right)^{1/2} \sum_p \langle[\nu]r|p|[\nu](m)\rangle p, \tag{2.13}$$

where $|[\nu](m)\rangle$ is a non-standard basis of $\mathrm{S}(f)$ to be decided below, and $\langle[\nu]r|p|[\nu](m)\rangle$ are the generalised matrix elements introduced by equation (102) in Chen and Gao (1982). Suppose

$$\left| \begin{matrix} [\nu] \\ r, (m) \end{matrix} \right\rangle = \mathscr{N} P_r^{[\nu](m)} |f_1 \ldots f_{m+n}\rangle, \tag{2.14}$$

$\mathscr{N}$ being a constant. From (2.3) we know that

$$\begin{aligned} p|f_1 \ldots f_{m+n}\rangle &= |f_1 \ldots f_{m+n}\rangle \qquad &&\text{for } p \in \prod_{i=1}^{m} \otimes \mathrm{S}(f_i), \\ q|f_1 \ldots f_{m+n}\rangle &= \delta_q |f_1 \ldots f_{m+n}\rangle \qquad &&\text{for } q \in \prod_{i=m+1}^{m+n} \otimes \mathrm{S}(f_i), \end{aligned} \tag{2.15}$$

where δ_q is the permutation parity of q. Combining (2.10)–(2.15), we obtain the eigenequations to be satisfied by the non-standard projection operator

$$\begin{pmatrix} C \\ C(s) \\ \bar{C}(s') \\ \bar{p} \\ \bar{q} \end{pmatrix} P_r^{[\nu](m)} = \begin{pmatrix} \nu \\ r \\ (m) \\ 1 \\ \delta_q \end{pmatrix} P_r^{[\nu](m)}. \tag{2.16}$$

From (2.10), (2.13) and (2.16) it is easy to show that the non-standard basis $|[\nu](m)\rangle$ obey the eigenequations

$$\begin{pmatrix} C \\ C(s') \\ p \\ q \end{pmatrix} |[\nu](m)\rangle = \begin{pmatrix} \nu \\ (m) \\ 1 \\ \delta_q \end{pmatrix} |[\nu](m)\rangle, \qquad \begin{array}{l} p \in \prod_{i=1}^{m} \otimes \mathrm{S}(f_i), \\ q \in \prod_{i=m+1}^{m+n} \otimes \mathrm{S}(f_i), \end{array} \tag{2.17a}$$

with

$$C(s') = (C(F_{N-1}) \ldots C(F_2), C(F_1)) \tag{2.17b}$$

where $C(F_i)$ is the CSCO-I of $\mathrm{S}(F_i)$. Let us prove one of the equations in (2.17a) as an example.

$$\bar{p}P_r^{[\nu](m)} = P_r^{[\nu](m)}p = P_r^{[\nu](m)} \tag{2.17c}$$

$$= \left(\frac{h_\nu}{f!}\right)^{1/2} \sum_{p_a} \langle[\nu]r|p_a|[\nu](m)\rangle p_a p$$

$$= \left(\frac{h_\nu}{f!}\right)^{1/2} \sum_{p_b} \langle[\nu]r|p_b p^{-1}|[\nu](m)\rangle p_b. \tag{2.17d}$$

Since (2.17d) holds for any Yamanouchi symbol r, it follows that

$$p^{-1}|[\nu](m)\rangle = |[\nu](m)\rangle.$$

This is just the third equation in (2.17a), because p^{-1} belongs to $\Pi_{i=1}^{m} \otimes \mathrm{S}(f_i)$.

Due to the fact that the CSCO-I of a finite group is the analogy of the Casmir operator set in the Lie group (Chen *et al* 1977a), equation (2.17a, b) shows that $|[\nu](m)\rangle$ is the IRB of $\mathrm{S}(f)$ adapted to the group chain

$$\mathrm{S}(f) \supset \boldsymbol{S}(f_1, f_2 \ldots f_N) \tag{2.18a}$$

$$\boldsymbol{S}(f_1, f_2 \ldots f_N) \equiv \mathrm{S}(F_{N-1}) \otimes \mathrm{S}(f_N) \supset \mathrm{S}(F_{N-2}) \otimes \mathrm{S}(f_{N-1}) \supset \ldots \supset \mathrm{S}(f_1) \otimes \mathrm{S}(f_2). \tag{2.18b}$$

Writing out all quantum numbers explicitly,

$$|[\nu](m)\rangle = \left| \begin{array}{c} [\nu] \to [\nu_{N-1}] \otimes [\tilde{f}_N] \\ \downarrow \\ \vdots \\ [\nu_m] \otimes [\tilde{f}_{m+1}] \\ \downarrow \\ [\nu_{m-1}] \otimes [f_m] \\ \downarrow \\ \vdots \\ [\nu_1] \otimes [\nu_2] \end{array} \right\rangle. \tag{2.19}$$

It belongs to the irrep $[\nu_{i-1}]$ of $S(F_{i-1})$, and to the totally symmetric irrep $[f_1]$ (antisymmetric irrep $[\tilde{f}_i] \equiv [1^{f_i}]$) of $S(f_i)$ for $i \leq m$ $(i > m)$. Obviously, the quantum numbers $[f_i]$ or $[\tilde{f}_i]$ are redundant, and the quantum number $[\nu](m)$ is sufficient for uniquely specifying the non-standard basis vector.

Now we turn to the determination of the constant $\mathcal{N}$ in (2.14). From the orthonormality of $|{}^{[\nu]}_{r,(m)}\rangle$, we have

$$|\mathcal{N}|^{-2} = \langle f_1 \ldots f_{m+n}| P_r^{[\nu](m)\dagger} P_r^{[\nu](m)} |f_1 \ldots f_{m+n}\rangle. \tag{2.20a}$$

We also have (Löwdin 1967)

$$P_r^{[\nu](m)\dagger} P_r^{[\nu](m)} = P_{(m)r}^{[\nu]} P_r^{[\nu](m)} = \left(\frac{f!}{h_\nu}\right)^{1/2} P_{(m)}^{[\nu](m)} = \sum_p \langle [\nu](m)|p|[\nu](m)\rangle p. \tag{2.21a}$$

Similarly

$$P_r^{[\nu](m)\dagger} P_r^{[\nu]s} = \sum_p \langle [\nu](m)|p|[\nu]s\rangle p, \tag{2.21b}$$

which is to be used in § 5.

Inserting (2.21a) into (2.20a),

$$|\mathcal{N}|^{-2} = \sum_p \langle [\nu](m)|p|[\nu](m)\rangle \langle f_1 \ldots f_{m+n}|p|f_1 \ldots f_{m+n}\rangle. \tag{2.20b}$$

Notice that, due to the orthonormality of the product states (2.3), only the permutations

$$p \in S(f_1 \ldots f_N), \qquad S(f_1 \ldots f_N) \equiv \prod_{i=1}^{N} \otimes S(f_i) \tag{2.22}$$

have non-vanishing contribution to (2.20a), and for those p we have either

$$p|[\nu](m)\rangle = |[\nu](m)\rangle, \qquad p|f_1 \ldots f_N\rangle = |f_1 \ldots f_N\rangle, \qquad \text{for } p \in \prod_{i=1}^{m} \otimes S(f_i),$$

or

$$p|[\nu](m)\rangle = \delta_p|[\nu](m)\rangle, \qquad p|f_1 \ldots f_N\rangle = \delta_p|f_1 \ldots f_N\rangle, \qquad \text{for } p \in \prod_{i=m+1}^{m+n} \otimes S(f_i). \tag{2.23}$$

Therefore (2.20b) is equal to the order of the group $S(f_1 \ldots f_N)$. Choosing $\mathcal{N}$ to be real positive, we get

$$\mathcal{N} = (f_1! f_2! \ldots f_{m+n}!)^{-1/2}. \tag{2.24}$$

Let us introduce the following notations:

$$|\{f_i\}\rangle = (f_i!)^{-1/2}|f_i\rangle, \tag{2.25}$$

$$|\Phi_0\rangle = |\{f_1\} \ldots \{f_{m+n}\}\rangle = \mathcal{N}|f_1 \ldots f_{m+n}\rangle. \tag{2.26a}$$

Thus

$$\left|\begin{matrix}[\nu] \\ r,(m)\end{matrix}\right\rangle = P_r^{[\nu](m)}|\Phi_0\rangle \tag{2.27}$$

are orthonormal, where $\{\Phi_0\}$ is orthogonal but not normalised,

$$\langle\Phi_0|\Phi_0\rangle = \mathcal{N}^2. \tag{2.26b}$$

It should be pointed out that the basis of $U(m)$ used by Sarma and Sahasrabudhe (1980) is not properly normalised, and they consequently have to introduce the factor $[(N_i+1)N_j]^{1/2}$ in their equation (9) *ad libitum*.

In summary, the Gel'fand basis of $U(m/n)$ can be constructed either through (2.8*b*) by using the standard projection operator of $S(f)$, where the norm $R^{(\nu)s}$ is to be decided, or through (2.27) by using the non-standard projection operator of $S(f)$, where the non-standard basis $|[\nu](m)\rangle$ is to be decided. A given Weyl tableau $W^{[\nu]}_{(m)}$ corresponds to a unique Gel'fand symbol $\binom{[\nu]}{(m)}$, which in turn specifies uniquely a non-standard basis vector $|[\nu](m)\rangle$ of $S(f)$. For example, for $m=n=2$,

$$W^{[\nu]}_{(m)} = \begin{array}{|c|c|c|c|c|} \hline a & a & b & \alpha & \beta \\ \hline b & b & \alpha \\ \cline{1-3} \alpha & \beta \\ \cline{1-2} \alpha \\ \cline{1-1} \end{array}, \qquad \begin{pmatrix}[\nu]\\(m)\end{pmatrix} = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 \\ & 3 & 2 & \\ & & 2 & \end{pmatrix},$$

$$|[\nu](m)\rangle = \left| \begin{array}{c} [5321]-[431^2]\otimes[1^2] \\ \downarrow \\ [32]\otimes[1^4] \\ \downarrow \\ [2]\otimes[3] \end{array} \right\rangle .$$

3. Decomposition of non-standard basis of $S(f)$

3.1. New symbol for the Gel'fand basis of $U(m/n)$

In § 2, we use (2.7) to label a Gel'fand basis of $U(m/n)$. This symbol is a straightforward extension of the ordinary Gel'fand basis; however, it is not convenient for what follows. From now on, we use the new symbol

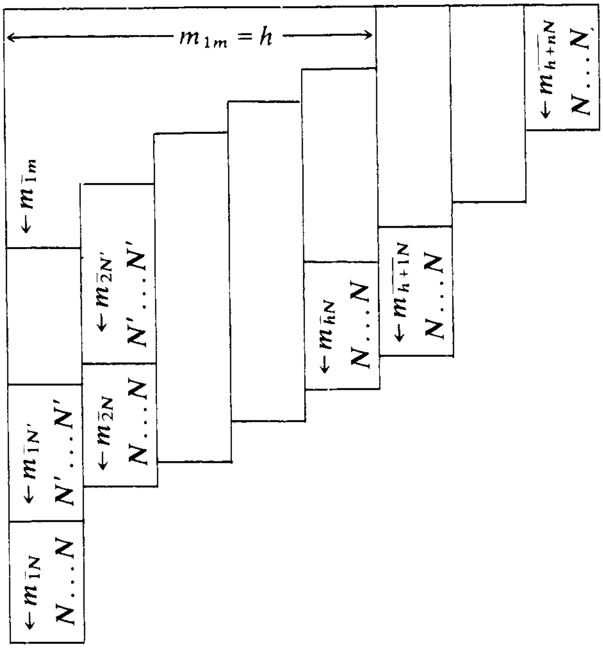
$$\begin{pmatrix}[\nu]\\(m)\end{pmatrix} \equiv \begin{pmatrix} [\tilde{\nu}] \\ \vdots \\ [\tilde{\nu}_{m+1}] \\ [\nu_m] \\ \vdots \\ [\nu_1] \end{pmatrix} = \begin{pmatrix} m_{\bar{1}m+n} \ldots m_{\overline{h+n},m+n} \\ \ldots \\ m_{\bar{1}m+1} \ldots m_{\overline{h+1},m+1} \\ m_{1m} \cdots m_{mm} \\ \ldots \\ m_{11} \end{pmatrix} \tag{3.1}$$

to label a Gel'fand basis of $U(m/n)$, where

$$h \equiv m_{1m}, \qquad [m_{\bar{i}k}] \equiv [m_{\bar{1}k}\, m_{\bar{2}k} \ldots m_{\overline{h+k},k}] = [\tilde{m}_{ik}] \equiv [\tilde{\nu}_k]. \tag{3.2}$$

The integer $m_{\bar{i}k}$ denotes the length of the ith column in the Weyl tableau $W^{[\nu]}_{(m)}$ after deleting all the boxes associated with the states $k+1, k+2, \ldots, m+n$, as is shown below:

$$W^{[\nu]}_{(m)} =$$



(3.3)

where $N' = N - 1$. For example

$$W^{(\nu)}_{(m)} = \begin{array}{|c|c|c|c|c|} \hline a & a & b & \alpha & \beta \\ \hline b & b & \alpha \\ \cline{1-3} \alpha & \beta \\ \cline{1-2} \alpha \\ \cline{1-1} \end{array}, \qquad \begin{pmatrix} [\nu] \\ (m) \end{pmatrix} = \begin{pmatrix} 4 \quad 3 \quad 2 \quad 1 \quad 1 \\ 4 \quad 2 \quad 2 \quad 1 \\ \hdashline 3 \quad 2 \\ 2 \end{pmatrix} \tag{3.4}$$

where the upper part above the broken line refers to $\left(\begin{smallmatrix}[m_{\bar{i}m+n}] \\ \vdots \\ [m_{\bar{i}m+1}]\end{smallmatrix}\right)$ whereas the lower part refers to $\left(\begin{smallmatrix}[m_{im}] \\ \vdots \\ [m_{11}]\end{smallmatrix}\right)$.

The integers m_{ij} have to satisfy the so-called betweenness condition (Baird and Biedenharn 1963)

$$\begin{array}{ccc} m_{ij+1} & \geqslant & m_{i+1,j+1} \\ & \searrow\!\!\!\!\geqslant \quad \geqslant\!\!\!\!\nearrow & \\ & m_{ij} & \end{array} \qquad \text{for } i, j \leqslant m-1. \tag{3.5a}$$

From (3.2) and (3.5a) we get the betweenness condition for $m_{\bar{i}j}$,

$$\begin{array}{ccc} m_{\bar{i}j+1} & \geqslant & m_{i+1,j+1} \\ & \searrow\!\!\!\!\geqslant \quad \geqslant\!\!\!\!\nearrow & \\ & m_{\bar{i}j}. & \end{array} \tag{3.5b}$$

We also have the restrictions that

$$m_{ij} = 0 \qquad \text{for } i > j+1, \tag{3.6a}$$

$$m_{\bar{i},m+k} = 0 \qquad \text{for } i > h+k+1 = m_{1m}+k+1. \tag{3.6b}$$

Equation (3.6a) comes from the fact that with j boson states, the maximum row number is j, while (3.6b) comes from the fact that with k fermion states, the maximum column number is equal to $h+k=m_{1m}+k$, as can be seen clearly from (3.3).

In the following, we will often use

$$f_{ik}=m_{ik}-m_{ik-1} \tag{3.7a}$$

to denote the number of the bosonic state k in the ith row of $W^{[\nu]}_{(m)}$, and use

$$f_{\bar{i}\kappa}=m_{\bar{i}\kappa}-m_{\bar{i}k-1} \tag{3.7b}$$

to denote the number of the fermionic state κ in the ith column of $W^{[\nu]}_{(m)}$. It follows from (3.6) that

$$f_{mm}=m_{mm}, \qquad f_{\overline{h+n},m+n}=m_{\overline{h+n},m+n}. \tag{3.8}$$

The total numbers of the bosonic state k and the fermionic state $m+k$ are given by

$$\begin{aligned} f_k &= \sum_i{}_1^k f_{ik}, \qquad & k&=1,2,\dots m, \\ f_{m+k} &= \sum_i{}_1^{h+k} f_{\bar{i},m+k}, \qquad & k&=1,2,\dots n, \end{aligned} \tag{3.9}$$

respectively.

3.2. The extended Yamanouchi symbol

The symbol (2.19) for the non-standard basis of S(f) is rather awkward. An elegant symbol, the extension of the Yamanouchi symbol, was introduced by Sarma and Saharasbudhe (1980) instead. We now extend it further to the more general case.

The basis vector of (2.19) can be represented by the following $(m+n)$ parentheses:

$$\begin{aligned} |[\nu](m)\rangle = {} & (1^{f_{11}})(1^{f_{12}}2^{f_{22}})\dots(1^{f_{1m}}\dots m^{f_{mm}}) \\ & \times((\overline{h+n})^{f_{\overline{h+n},m+n}}\dots\bar{1}^{f_{\bar{1}m+n}})\dots((\overline{h+1})^{f_{\overline{h+1},m+1}}\dots\bar{1}^{f_{\bar{1}m+1}}). \end{aligned} \tag{3.10a}$$

where the kth parenthesis

$$(1^{f_{1k}}\dots i^{f_{ik}}\dots k^{f_{kk}}) \tag{3.10b}$$

signifies a basis which is totally symmetric with respect to the permutation group S(f_k) associated with the particle $F_{k-1}+1, F_{k-1}+2,\dots,F_k$, and in which there are f_{ik} particles (f_{ik} states χ_k) in the ith row of the Young diagram $Y^{(\nu)}$ (Weyl tableau $W^{[\nu]}_{(m)}$), $i=1,2,\dots,k$. Similarly, the $(m+k)$th parenthesis

$$((\overline{h+k})^{f_{\overline{h+k},m+k}}\dots\bar{j}^{f_{\bar{j},m+k}}\dots\bar{1}^{f_{\bar{1},m+k}}) \tag{3.10c}$$

signifies a basis which is totally antisymmetric with respect to S(f_{m+k}), and in which there are $f_{\bar{j},m+k}$ particles ($f_{\bar{j},m+k}$ states ψ_{m+k}) in the jth column of $Y^{(\nu)}$ (Weyl tableau $W^{[\nu]}_{(m)}$). For example, the non-standard basis of S(11) corresponding to the Weyl tableau in (3.4) is now designated as

$$|[\nu](m)\rangle=(1^2)(12^2)(\bar{4}\bar{3}\bar{1}^2)(\bar{5}\bar{2}). \tag{3.11}$$

In the case when each parenthesis in (3.10a) contains either only one row number or only one column number, the extended Yamanouchi symbol goes back to the Yamanouchi symbol $(r_1\dots r_f)$ (noting that it differs from the definition used in Chen *et al* (1983b) where, following Hamermesh (1962), $(r_f \ldots r_1)$ is defined as the Yamanouchi symbol). For instance

$$W^{[\nu]}_{(m)} = \begin{array}{llll} a & a & a & \beta \\ b & b & \gamma & \\ \alpha & & & \\ \alpha & & & \end{array}, \qquad |[\nu](m)\rangle = (1^3)(2^2)(\bar{1}^2)(\bar{4})(\bar{3}) = \begin{array}{llll} 1 & 2 & 3 & 8 \\ 4 & 5 & 9 & \\ 6 & & & \\ 7 & & & \end{array} \tag{3.12}$$

Evidently we have

$$(j)^n = (j^n), \qquad (\bar{j})^n = (\bar{j}^n). \tag{3.13}$$

3.3. *The construction of* $|[\nu](m)\rangle$

Suppose Y_r and Y_s are the two Young tableaux which differ only by an interchange of the positions of the particles i and $i+1$, and the Yamanouchi symbol $r<s$ (or $r>s$ in Hamermesh's definition of the Yamanouchi symbol). Then

$$\Psi^{[2]}(i, i+1) = [(\sigma-1)/(2\sigma)]^{1/2} Y_r + [(\sigma+1)/(2\sigma)]^{1/2} Y_s \tag{3.14a}$$

is symmetric in the indices i and $i+1$, while

$$\Psi^{[11]}(i, i+1) = [(\sigma+1)/(2\sigma)]^{1/2} Y_r - [(\sigma-1)/(2\sigma)]^{1/2} Y_s \tag{3.14b}$$

is antisymmetric in i and $i+1$ (Jahn 1954), where $\sigma>0$ is the axial distance between i and $i+1$. By repeatedly using (3.14) for all pairs $(i, i+1)$ which are to be symmetrised or antisymmetrised, we can construct the non-standard basis $|[\nu](m)\rangle$ out of the standard basis.

For instance, $|[\nu](m)\rangle = (1^3)(2^2)(\bar{4}\bar{3}\bar{1})$ can be constructed out of the following six Yamanouchi basis vectors:

$$\begin{array}{cccccc}
(\bar{4})(\bar{3})(\bar{1}) & (\bar{4})(\bar{1})(\bar{3}) & (\bar{3})(\bar{4})\bar{1}) & (\bar{3})(\bar{1})(\bar{4}) & (\bar{1})(\bar{4})(\bar{3}) & (\bar{1})(\bar{3})(\bar{4}) \\
\begin{array}{|c|c|c|c|}\hline & & & 6 \\ \hline & & 7 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline & & & 6 \\ \hline & & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline & & & 7 \\ \hline & & 6 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline & & & 8 \\ \hline & & 6 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline & & & 7 \\ \hline & & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline & & & 8 \\ \hline & & 7 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array} \\
|1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle & |6\rangle,
\end{array} \tag{3.15a}$$

where we deliberately leave the Young diagram [32] blank instead of filling it with numbers, and to the top of each Young tableau we attached the symbol $(\bar{i})(\bar{j})(\bar{k})$ in accordance with (3.10), which specify the column numbers of the particles 6, 7 and 8, respectively. For brevity, we write $|[32](\bar{4}\bar{3}\bar{1})\rangle$ for $(1^3)(2^2)(\bar{4}\bar{3}\bar{1})$, and $|i\rangle$ for $|Y_i\rangle$. Thus

$$|[32](\bar{4}\bar{3}\bar{1})\rangle = \sum_i{}_1^6 a_i|i\rangle. \tag{3.15b}$$

From the requirement that $(\bar{4}\bar{3}\bar{1})$ is antisymmetric in the particles 6 and 7, we get by using (3.14*b*) that

$$a_1/a_3 = -\sqrt{3}, \qquad a_2/a_5 = -\sqrt{\tfrac{3}{2}}, \qquad a_4/a_6 = -\sqrt{2}. \tag{3.16a}$$

From $(\bar{4}\bar{3}\bar{1})$ being antisymmetric in 7 and 8, we have

$$a_1/a_2 = -\sqrt{2}, \qquad a_3/a_4 = -\sqrt{\tfrac{3}{2}}, \qquad a_5/a_6 = -\sqrt{3}. \tag{3.16b}$$

After normalisation, from (3.16) we obtain

$$|[32](\bar{4}\bar{3}\bar{1})\rangle = \sqrt{\tfrac{27}{45}}(\sqrt{\tfrac{2}{3}}|1\rangle - \sqrt{\tfrac{1}{3}}|2\rangle) - \sqrt{\tfrac{10}{45}}(\sqrt{\tfrac{3}{5}}|3\rangle - \sqrt{\tfrac{2}{5}}|4\rangle) + \sqrt{\tfrac{8}{45}}(\sqrt{\tfrac{3}{4}}|5\rangle - \sqrt{\tfrac{1}{4}}|6\rangle) \qquad (3.17)$$

Notice that the combination coefficients a_i only depend on the axial distances between the numbers 6, 7 and 8, and are totally independent of how numbers are filled in the Young diagram [32]. Therefore, (3.17) remains true under the substitutions

$$(1^3)(2^2)(\bar{4}\bar{3}\bar{1}) \to (1^2)(12^2)(\bar{4}\bar{3}\bar{1}),\ (1^3)(2^2)(\bar{i})(\bar{j})(\bar{k}) \to (1^2)(12^2)(\bar{i})(\bar{j})(\bar{k}).$$

This is the reason why we leave the Young diagram [32] in (3.15a) blank.

The above observation can be epitomised by the *parenthesis independent decomposition rule*: each parenthesis in $|[\nu](m)\rangle$ can be decomposed into (or constructed out of) the Yamanouchi basis independently.

This rule is crucial for later developments and should be well kept in mind.

It is easy to see that the following combination is antisymmetric in i and $i+1$:

$$\Phi^{[11]}(i, i+1) = -[(\sigma-1)/(2\sigma)]^{1/2} Y_{\tilde{r}} + [(\sigma+1)/)2\sigma)]^{1/2} Y_{\tilde{s}} \qquad (3.18)$$

where $Y_{\tilde{r}} \equiv \tilde{Y}_r$ and $Y_{\tilde{s}} \equiv \tilde{Y}_s$ are the cojugated tableaux of Y_r and Y_s, and $\tilde{s} < \tilde{r}$ if $r < s$. Comparing (3.18) with (3.14a), it is clear that the antisymmetric combination coefficient in front of $Y_{\tilde{r}}$ is equal to the symmetric combination coefficient in front of Y_r up to a sign factor. We call it the *symmetry of the combination coefficients under conjugation*. By this symmetry, once the symmetric combination coefficients are known, the antisymmetric combination coefficients are easily obtainable by properly considering the sign factors, and *vice versa*. For instance, from (3.17), we immediately get the basis vector which is symmetric in 6, 7 and 8,

$$\begin{aligned} |[\widetilde{32}](134)\rangle &= |[221](134)\rangle \\ &= \sqrt{\tfrac{27}{45}}(\sqrt{\tfrac{2}{3}}|\tilde{1}\rangle + \sqrt{\tfrac{2}{3}}|\tilde{2}\rangle) + \sqrt{\tfrac{10}{45}}(\sqrt{\tfrac{3}{5}}|\tilde{3}\rangle + \sqrt{\tfrac{2}{5}}|\tilde{4}\rangle) + \sqrt{\tfrac{8}{45}}(\sqrt{\tfrac{3}{4}}|\tilde{5}\rangle + \sqrt{\tfrac{1}{4}}|\tilde{6}\rangle). \end{aligned} \qquad (3.19)$$

where $|\tilde{i}\rangle$ stands for $|\tilde{Y}_i\rangle$.

3.4. *The decomposition of* $|[\nu](m)\rangle$

Due to the parenthesis independent decomposition rule, each parenthesis of $|(\nu)(m)\rangle$ in (3.10a) can be tackled individually. The decomposition of the symmetric parentheses is already known. According to Sarma and Saharasbudhe (1980), the left and right decompositions of the kth symmetric parenthesis are

$$(1^{f_{1k}} \ldots j^{f_{jk}} \ldots k^{f_{kk}}) = \sum_{j=1}^{k}{}' a_j^{(k)} (j)(\ldots j^{f_{jk}-1} \ldots), \qquad (3.20a)$$

$$(1^{f_{1k}} \ldots j^{f_{jk}} \ldots k^{f_{kk}}) = \sum_{j=1}^{k}{}' b_j^{(k)} (\ldots j^{f_{jk}-1} \ldots)(j), \qquad (3.20b)$$

with the normalised combination coefficients

$$a_j^{(k)} = \left(\frac{\prod_{i=1}^{k} (m_{ik} - m_{jk-1} - i + j)}{\prod_{1, i \neq j}^{k} (m_{ik-1} - m_{jk-1} - i + j)} \right)^{1/2} f_k^{-1/2}, \qquad (3.21a)$$

$$b_j^{(k)}=(-1)^{1/2}\left(\frac{\prod_{i=1}^{k}(m_{ik-1}-m_{jk}+j-i)}{\prod_{\substack{1\\ i\neq j}}^{k}(m_{ik}-m_{jk}+j-i)}\right)^{1/2} f_k^{-1/2}. \tag{3.21b}$$

The prime in (3.20) indicates that the summations are restricted to those indices j for which $f_{jk}\geqslant 1$.

In order to decompose the antisymmetric $(m+k)$th parenthesis

$$(A)\equiv((\overline{h+k})^{f_{\overline{h+k},m+k}}\ldots\bar{j}^{f_{\bar{j},m+k}}\ldots\bar{1}^{f_{\bar{1},m+k}}), \tag{3.22}$$

we turn to its conjugated (symmetric) basis

$$(\tilde{A})\equiv(1^{f_{\bar{1},m+k}}\ldots j^{f_{\bar{j},m+k}}\ldots(h+k)^{f_{\overline{h+k},m+k}}). \tag{3.23}$$

Comparing (3.23) with the left of (3.20), we see that there is a slight difference between the two. In (3.20), the maximum row number for the state k is equal to k, whereas in (3.23) the maximum row number (i.e. the maximum column number in the origninal Weyl tableau $W_{(m)}^{[\nu]}$) for the state $m+k$ is equal to $h+k$, and in general $m\neq h$. However, we are free to rename the $(m+k)$th state as the state $h+k$, i.e. let $\psi_{m+k}=\psi'_{h+k}$, or $(m+k)=(h+k)'$, and then use (3.21) to obtain the decomposition coefficients for (3.23), and finally switch back to the old name by letting $(h+k)'\to(m+k)$. In this way we have

$$(\tilde{A})=\sum_{j=1}^{h+k}{}'\,\tilde{a}_j^{(m+k)}(j)(\ldots j^{f_{\bar{j},m+k}-1}\ldots), \tag{3.24a}$$

$$(\tilde{A})=\sum_{j=1}^{h+k}{}'\,\tilde{b}_j^{(m+k)}(\ldots j^{f_{\bar{j},m+k}-1}\ldots)(j), \tag{3.24b}$$

$$\tilde{a}_j^{(m+k)}=\left(\frac{\prod_{i=1}^{h+k}(m_{\bar{i},m+k}-m_{\bar{j},m+k-1}-i+j)}{\prod_{\substack{1\\ i\neq j}}^{h+k}(m_{\bar{i},m+k-1}-m_{\bar{j},m+k-1}-i+j)}\right)^{1/2}(f_{m+k})^{-1/2}, \tag{3.25a}$$

$$\tilde{b}_j^{(m+k)}=(-1)^{1/2}\left(\frac{\prod_{i=1}^{h+k}(m_{\bar{i},m+k-1}-m_{\bar{j},m+k}+j-i)}{\prod_{\substack{1\\ i\neq j}}^{h+k}(m_{\bar{i},m+k}-m_{\bar{j},m+k}+j-i)}\right)^{1/2}(f_{m+k})^{-1/2}. \tag{3.25b}$$

The prime in (3.24a) and (3.24b) restricts the summation index j to those for which $f_{\bar{j},m+k}\geqslant 1$.

Let the $(m+k)$th antisymmetric parenthesis be decomposed as

$$(A)=\sum_{j=1}^{h+k}{}'\,a_{\bar{j}}^{(m+k)}(\bar{j})(\ldots\bar{j}^{f_{\bar{j},m+k}-1}\ldots), \tag{3.26a}$$

$$(A)=\sum_{j=1}^{h+k}{}'\,b_{\bar{j}}^{(m+k)}(\ldots\bar{j}^{f_{\bar{j},m+k}-1}\ldots)(\bar{j}). \tag{3.26b}$$

According to the symmetry of the combination coefficients under conjugation, we have

$$a_{\bar{j}}^{(m+k)}=\varepsilon_j^{(m+k)}\tilde{a}_j^{(m+k)},\qquad b_{\bar{j}}^{(m+k)}=\eta_j^{(m+k)}\tilde{b}_j^{(m+k)}, \tag{3.27a, b}$$

where ε and η are sign factors, Now we claim that

$$\varepsilon_j^{(m+k)} = (-1)^{\sum_{i\,j+1}^{h+k} f_{\bar{i},m+k}}, \qquad \eta_j^{(m+k)} = (-1)^{\sum_{i\,1}^{j-1} f_{\bar{i},m+k}}. \qquad (3.28a, b)$$

To justify (3.28a), let us decompose the $(m+k)$th parenthesis twice. Assuming $j>i$,

$$(\overline{h+k})^{f_{\overline{h+k},m+k}} \ldots \bar{j}^{f_{\bar{j},m+k}} \ldots \bar{i}^{f_{\bar{i},m+k}} \ldots)$$

$$= \sum_{ij}{}_1^{h+k}{}' [a_{\bar{j}} a'_{\bar{i}} (\bar{j})(\bar{i}) + a_{\bar{i}} a'_{\bar{j}} (\bar{i})(\bar{j})](\ldots \bar{j}^{f_{\bar{j},m+k}-1} \ldots \bar{i}^{f_{\bar{i},m+k}-1} \ldots), \qquad (3.29)$$

where $a_{\bar{i}}, a_{\bar{j}}$ and $a'_{\bar{i}}, a'_{\bar{j}}$ are the decomposition coefficients for the first and second times respectively. On the basis of (3.28a), we have

$$\operatorname{sgn}(a_{\bar{j}} a'_{\bar{i}}) = (-1)^{f_{\bar{j},m+k}-1} \cdot \zeta, \qquad \operatorname{sgn}(a_{\bar{i}} a'_{\bar{j}}) = (-1)^{f_{\bar{j},m+k}} \cdot \zeta,$$

where ζ is another sign factor. Therefore

$$\operatorname{sgn}(a_{\bar{j}} a'_{\bar{i}} / a_{\bar{i}} a'_{\bar{j}}) = -1.$$

According to (3.18), this minus sign ensures that the right-hand side of (3.29) is antisymmetric in the particles $F_{m+k-1}+1$ and $F_{m+k-1}+2$. Equation (3.28a) can be similarly justified.

Assembling (3.25), (3.27) and (3.28), we finally get the decomposition coefficients for the antisymmetric parenthesis:

$$a_{\bar{j}}^{(\kappa)} = (-1)^{\sum_i^{h+k} (m_{\bar{i}\kappa} - m_{\bar{i}\kappa-1})} \left(\frac{\prod_{i=1}^{h+k} (m_{\bar{i}\kappa} - m_{\bar{j}\kappa-1} - i + j)}{\prod_{1, i \neq j}^{h+k} (m_{\bar{i}\kappa-1} - m_{\bar{j}\kappa-1} - i + j)} \right)^{1/2} (f_\kappa)^{-1/2}, \qquad (3.30a)$$

$$b_{\bar{j}}^{(\kappa)} = (-1)^{1/2} (-1)^{\sum_i^{j-1} (m_{\bar{i}\kappa} - m_{\bar{i}\kappa-1})} \left(\frac{\prod_{i=1}^{h+k} (m_{\bar{i}\kappa-1} - m_{\bar{j}\kappa} + j - i)}{\prod_{1, i \neq j}^{h+k} (m_{\bar{i}\kappa} - m_{\bar{j}\kappa} + j - i)} \right)^{1/2} (f_\kappa)^{-1/2}, \qquad (3.30b)$$

where

$$\kappa = m + k, \qquad (m_{\bar{i}m}) = (\tilde{m}_{im}). \qquad (3.30c)$$

3.5. *The special case of* U(2/1)

Consider the most general Gel'fand basis vector of U(2/1), the Weyl tableau of which looks like

1	2	...	p	...	q
a	a	a	b	b	α
b	b	b	α		
α					
α					
α					

$$p = m_{22} + 1, \qquad q = h + 1 = m_{12} + 1 \qquad (3.31)$$

where we introduced the two integers p and q.

It is clear from (3.31) that

$$m_{\bar{p}2} = m_{\bar{q}3} = 1, \qquad m_{\bar{p}3} = m_{\bar{1}2} = 2, \qquad f_3 = m_{\bar{1}3}. \qquad (3.32)$$

There is only one antisymmetric parenthesis $(\bar q\bar p\bar 1^{m_{\bar 13}-2})$. Using (3.26*a*), (3.30*a*) and (3.32), it can be decomposed as

$$\begin{aligned}(\bar q\bar p\bar 1^{m_{\bar 13}-2}) = {} & \left(\frac{(m_{12}-m_{22}+2)(m_{\bar 13}+m_{12})}{(m_{12}-m_{22}+1)(m_{12}+2)m_{\bar 13}}\right)^{1/2}(\bar q)(\bar p\bar 1^{m_{\bar 13}-2})\\ & -\left(\frac{(m_{12}-m_{22})(m_{\bar 13}+m_{22}-1)}{(m_{12}-m_{22}+1)(m_{22}+1)m_{\bar 13}}\right)^{1/2}(\bar p)(\bar q\bar 1^{m_{\bar 13}-2})\\ & +\left(\frac{(m_{12}+1)m_{22}(m_{\bar 13}-2)}{(m_{12}+2)(m_{22}+1)m_{\bar 13}}\right)^{1/2}(\bar 1)(\bar q\bar p\bar 1^{m_{\bar 13}-3}).\end{aligned}\tag{3.33}$$

As a verification of (3.33), let us assume that

$$W^{[\nu]}_{(m)}=\begin{array}{|c|c|c|c|}\hline a & a & b & \alpha\\\hline b & b & \alpha & \\\cline{1-3} \alpha & & & \\\cline{1-1}\end{array}\,,\qquad \begin{pmatrix}[\nu]\\(m)\end{pmatrix}=\begin{pmatrix}m_{\bar 13}m_{\bar 23}m_{\bar 33}m_{\bar 43}\\ m_{12}m_{22}\\ m_{11}\end{pmatrix}=\begin{pmatrix}3\;\;2\;\;2\;\;1\\ 3\;\;2\\ 2\end{pmatrix},$$

$$[m_{\bar 12}m_{\bar 22}m_{\bar 32}]=[\widetilde{32}]=[221].$$

According to (3.33), we have the decomposition

$$(\bar 4\bar 3\bar 1)=\sqrt{\tfrac{27}{45}}(\bar 4)(\bar 3\bar 1)-\sqrt{\tfrac{10}{45}}(\bar 3)(\bar 4\bar 1)+\sqrt{\tfrac{8}{45}}(\bar 1)(\bar 4\bar 3)\tag{3.34}$$

which is totally in agreement with equation (3.17) obtained through direct antisymmetrisation procedure.

4. The matrix elements of generators $E^{i-1}{}_i$ of U(m/n)

4.1. *Matrix elements of $E^{i-1}{}_i$ and overlaps between $|[\nu](m)\rangle$*

We first derive a general formula relating the Gel'fand matrix elements of $E^{i-1}{}_i$ with the overlap integrals between the non-standard basis vector of S(f), i.e.

$$\left\langle\begin{pmatrix}[\nu]\\(m')\end{pmatrix}\middle|E^{i-1}{}_i\middle|\begin{pmatrix}[\nu]\\(m)\end{pmatrix}\right\rangle=[(f_{i-1}+1)f_i]^{1/2}\langle[\nu](m')|[\nu](m)\rangle.\tag{4.1}$$

Let us prove it separately in the following three cases.

(i) *Boson–boson case*

In this case, both i and $i-1$ refer to the bosonic states. Using (2.27) and (2.25) as well as the commutativity of $E^{i-1}{}_i$ with $P_r^{[\nu](m)}$, we have

$$\begin{aligned}E^{i-1}{}_i\left|\begin{matrix}[\nu]\\ r,(m)\end{matrix}\right\rangle &= P_r^{[\nu](m)}E^{i-1}{}_i|\{f_1\}\ldots\{f_{m+n}\}\rangle\\ &=(f_i!)^{-1/2}P_r^{[\nu](m)}|\{f_1\}\ldots(\chi_{i-1}\chi_i^{f_i-1})\ldots\{f_{m+n}\}\rangle,\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}(\chi_{i-1}\chi_i^{f_i-1}) &= \sum_{k=1}^{f_i}|\chi_i^{1'}\ldots\chi_i^{k'-1}\chi_{i-1}^{k'}\chi_i^{k'+1}\ldots\chi_i^{F_i'}\rangle\\ &=\sum_{k=1}^{f_i}(1'2'\ldots k')^{-1}|\chi_{i-1}^{1'}\chi_i^{2'}\ldots\chi_i^{F_i'}\rangle,\end{aligned}\tag{4.3a}$$

$$k' = F_{i-1} + k. \tag{4.3b}$$

In (4.3*a*), $(1'2'\ldots k')^{-1}$ is the inverse of the k-cycle permutation operator.

Inserting (4.3*a*) into (4.2) and making use of (2.25),

$$E^{i-1}{}_i \left| \begin{matrix}[\nu] \\ r, (m)\end{matrix} \right\rangle = [(f_{i-1}+1)/f_i]^{1/2} P_r^{[\nu](m)} \sum_{k=1}^{f_i} (1'2'\ldots k')|\Phi_0'\rangle$$
$$= [(f_{i-1}+1)f_i]^{1/2} P_r^{[\nu](m)} |\Phi_0'\rangle, \tag{4.4}$$

where (2.17*c*) has been used, and

$$|\Phi_0'\rangle = |\{f_1\}\ldots\{f_{i-1}+1\}\{f_i-1\}\ldots\{f_{m+n}\}\rangle. \tag{4.5}$$

From (4.4), (2.21*b*) and (2.27) we have

$$\left\langle \begin{pmatrix}[\nu] \\ (m')\end{pmatrix} \right| E^{i-1}{}_i \left| \begin{pmatrix}[\nu] \\ (m)\end{pmatrix} \right\rangle = \left\langle \begin{matrix}[\nu] \\ r, (m')\end{matrix} \right| E^{i-1}{}_i \left| \begin{matrix}[\nu] \\ r, (m)\end{matrix} \right\rangle$$
$$= [(f_{i-1}+1)f_i]^{1/2} \sum_p \langle [\nu](m')|p|[\nu](m)\rangle \langle \Phi_0'|p|\Phi_0'\rangle. \tag{4.6}$$

Using the fact that only the permutations p belonging to the subgroup $S(f_1, \ldots f_{i-1}+1, f_i-1, \ldots f_{m+n})$ have non-zero contribution to (4.6), and that for those p we have

$$p^{-1}|[\nu](m')\rangle = |[\nu](m')\rangle, \qquad p|\Phi_0'\rangle = |\Phi_0'\rangle,$$
$$\text{for } p \in S(f_1) \otimes \ldots S(f_{i-1}+1) \otimes S(f_i-1) \otimes \ldots S(f_m), \tag{4.7a}$$

$$p^{-1}|[\nu](m') = \delta_p |[\nu](m')\rangle, \qquad p|\Phi_0'\rangle = \delta_p|\Phi_0'\rangle,$$
$$\text{for } p \in S(f_{m+1}) \otimes \ldots \otimes S(f_{m+n}), \tag{4.7b}$$

equation (4.6) can be identified with (4.1).

(ii) *Fermion–boson case*

Now $E^m{}_{m+1}$ represents an operator which shift the fermionic state $m+1$ into the bosonic state m. The analogy of (4.2) is

$$E^m{}_{m+1} \left| \begin{matrix}[\nu] \\ r, (m)\end{matrix} \right\rangle = (f_{m+1}!)^{-1/2} P_r^{[\nu](m)} |\{f_1\}\ldots(\chi_m \psi_{m+1}^{f_i-1})\ldots\{f_{m+n}\}\rangle, \tag{4.8}$$

$$(\chi_m \psi_{m+1}^{f_i-1}) \equiv \sum_{k=1}^{f_{m+1}} (-1)^{k-1} |\psi_{m+1}^{1'} \ldots \psi_{m+1}^{k'-1} \chi_m^{k'} \psi_{m+1}^{k'+1} \ldots \psi_{m+1}^{F_{m+1}}\rangle$$
$$= \sum_{k}{}_1^{f_{m+1}} (-1)^{k-1} (1'2'\ldots k')^{-1} |\chi_m^{1'} \psi_{m+1}^{2'} \ldots \psi_{m+1}^{F_{m+1}}\rangle, \tag{4.9}$$

where $k' = F_m + k$. Since $(1'2'\ldots k')^{-1}$ belongs to $S(F_{m+1})$, from (2.16) and (2.10) we have

$$P_r^{[\nu](m)} (1'2'\ldots k')^{-1} = (-1)^{k-1} P_r^{[\nu](m)}. \tag{4.10}$$

Combining (4.8)–(4.10) leads to

$$E^m{}_{m+1} \left| \begin{matrix}[\nu] \\ r, (m)\end{matrix} \right\rangle = [(f_m+1)f_{m+1}]^{1/2} P_r^{[\nu](m)} |\{f_1\}\ldots\{f_m+1\}\{f_{m+1}-1\}\ldots\{f_{m+n}\}\rangle. \tag{4.11}$$

Therefore we again have

$$\left\langle\begin{pmatrix}[\nu]\\(m')\end{pmatrix}\middle|E^{m}{}_{m+1}\middle|\begin{pmatrix}[\nu]\\(m)\end{pmatrix}\right\rangle=[(f_m+1)f_{m+1}]^{1/2}\langle[\nu](m')|[\nu](m)\rangle. \tag{4.12}$$

(iii) *Fermion–fermion case*

For $i \geqslant m+2$, we have

$$E^{i-1}{}_i\left|\begin{matrix}[\nu]\\r,(m)\end{matrix}\right\rangle=(f_i!)^{-1/2}P_r^{[\nu](m)}|\{f_1\}\ldots(\psi_{i-1}\psi_i^{f_i-1})\ldots\{f_{m+n}\}\rangle, \tag{4.13}$$

$$(\psi_{i-1}\psi_i^{f_i-1})=\sum_k{}_1^{f_i}|\psi_i^{1'}\ldots\psi_i^{k'-1}\psi_{i-1}^{k'}\psi_i^{k'+1}\ldots\psi_i^{F_i}\rangle$$

$$=\sum_k{}_1^{f_i}(-1)^{k-1}(1'2'\ldots k')^{-1}|\psi_{i-1}^{1'}\psi_i^{2'}\ldots\psi_i^{F_i}\rangle, \tag{4.14}$$

where $k'=F_{i-1}+k$, and the sign factor $(-1)^{k-1}$ comes from the anticommutativity of ψ's. Using (4.10) and (4.14), equation (4.13) becomes

$$E^{i-1}{}_i\left|\begin{matrix}[\nu]\\r,(m)\end{matrix}\right\rangle=[(f_{i-1}+1)f_i]^{1/2}P_r^{[\nu](m)}|\{f_1\}\ldots\{f_m\}\ldots\{f_{i-1}+1\}\{f_i-1\}\ldots\rangle.$$

Hence equation (4.1) still holds for $i \geqslant m+2$.

4.2. Formulae for matrix elements of $E^{i-1}{}_i$

Equation (4.1) converts the problem of evaluating the matrix elements of $E^{k-1}{}_k$ into a much easier one, i.e. calculating the overlaps between the non-standard basis vectors of S(f), $|[\nu](m')\rangle$ and $|[\nu](m)\rangle$. Notice that they are not orthogonal, since $|[\nu](m)\rangle$ is adapted to the group chain

$$\mathrm{S}(f)\supset \boldsymbol{S}(f_1,\ldots f_{k-1},f_k,\ldots f_{m+n}) \tag{4.15a}$$

whereas $|[\nu](m')\rangle$ is adapted to the group chain

$$\mathrm{S}(f)\supset \boldsymbol{S}(f_1,\ldots f_{k-1}+1,f_k-1,\ldots f_{m+n}). \tag{4.15b}$$

In order to evaluate the overlap $\langle[\nu](m')|[\nu](m)\rangle$, we first decompose the non-standard basis $|[\nu](m)\rangle$ and $|[\nu](m')\rangle$ into the IRB adapted to the common group chain

$$\mathrm{S}(f)\supset \boldsymbol{S}(f_1,\ldots f_{k-1},1,f_k-1,\ldots f_{m+n}), \tag{4.15c}$$

and then use the orthonormality of the new basis to obtain the overlap. Evidently, for $|[\nu](m)\rangle$ we need to use the left decomposition, and for $|[\nu](m')\rangle$ the right decomposition. The matrix elements of $E^{k-1}{}_k$ are given below in the three cases.

(i) *Boson–boson*

Consider the case where a state k in the jth row of a Weyl tableau $W^{[\nu]}_{(m)}$ is being changed into the state $k-1$ under the operation of $E^{k-1}{}_k$. In order that the resulting Weyl tableau $W^{[\nu]}_{(m')}$ be lexical, this state k must lie at the leftmost position of the jth row. The matrix element for this case is

$$M^{(j)}_{k-1,k}(\text{B–B}) = \left\langle \begin{pmatrix} m_{\bar{1}m+n} \dots m_{\overline{h+n},m+n} \\ \dots \\ m_{1k} \dots m_{jk} \dots m_{kk} \\ m_{1k-1} \dots m_{jk-1}+1 \dots m_{k-1,k-1} \\ \dots \\ m_{11} \end{pmatrix} \middle| E^{k-1}{}_{k} \middle| \begin{pmatrix} m_{\bar{1}m+n} \dots m_{\overline{h+n},m+n} \\ \dots \\ m_{1k} \dots m_{jk} \dots m_{kk} \\ m_{1k-1} \dots m_{jk-1} \dots m_{k-1,k-1} \\ \dots \\ m_{11} \end{pmatrix} \right\rangle$$

$$= [(f_{k-1}+1)f_k]^{1/2} a_j^{(k)} b_j'^{(k-1)}, \tag{4.16a}$$

where

$$b_j'^{(k-1)} = b_j^{(k-1)}(m_{j,k-1} \to m_{j,k-1}+1). \tag{4.17}$$

Equation (4.16a) is identical with equation (29) of Sarma and Saharasbudhe (1980); however, here the factor $[(f_{k-1}+1)f_k]^{1/2}$ emerges naturally instead of being put in by hand.

Using (3.21), we get

$$M^{(j)}_{k-1,k}(\text{B–B}) = (-1)^{1/2} \left(\frac{\prod_{i=1}^{k}(m_{ik}-m_{jk-1}-i+j)\ \prod_{i=1}^{k-1}(m_{ik-2}-m_{jk-1}+j-i-1)}{\prod_{1, i\neq j}^{k}(m_{ik-1}-m_{jk-1}-i+j)\ \prod_{1, i\neq j}^{k-1}(m_{ik-1}-m_{jk-1}+j-i-1)} \right)^{1/2}, \tag{4.16b}$$

which is just the usual Gel'fand–Tsetlin formula.

(ii) *Fermion–boson*

Take up the case where a state $m+1$ in the $\bar{j}$th column of $W^{[\nu]}_{(m)}$ is being changed into m under the action of $E^{m}{}_{m+1}$. Analogously, for the resulted Weyl tableau $W^{[\nu]}_{(m')}$ being lexical, this state $m+1$ has to be at the topmost row, say tth row, of the $\bar{j}$th column. It implies that under the operation

$$[m_{im}] \to [m'_{im}] = [m_{1m} \dots m_{tm}+1 \dots],$$
$$[m_{\bar{i}m}] \to [m'_{\bar{i}m}] = [\tilde{m}'_{im}] = [m_{\bar{1}m} \dots m_{\bar{j}m}+1 \dots]. \tag{4.18}$$

Notice that a given $\bar{j}$ corresponds to a unique t. The matrix element for the above case is equal to

$$M^{(t,\bar{j})}_{m,m+1}(\text{F–B}) = \left\langle \begin{pmatrix} m_{\bar{1}m+n} \dots m_{\overline{h+n},m+n} \\ \dots \\ m_{\bar{1}m+1} \dots m_{\overline{h+1},m+1} \\ m_{1m} \dots m_{tm}+1 \dots m_{mm} \\ \dots \\ m_{11} \end{pmatrix} \middle| E^{m}_{m+1} \middle| \begin{pmatrix} m_{\bar{1}m+n} \dots m_{\overline{h+n},m+n} \\ \dots \\ m_{\bar{1}m+1} \dots m_{\overline{h+1},m+1} \\ m_{1m} \dots m_{tm} \dots m_{mm} \\ \dots \\ m_{11} \end{pmatrix} \right\rangle$$

$$= [(f_m+1)f_{m+1}]^{1/2} \langle [\nu](m') \| [\nu](m) \rangle. \tag{4.19a}$$

From (3.10*a*), (4.18), (3.26*a*) and (3.20*b*), we have

$$|[\nu](m)\rangle = (1^{f_{1m}}\dots t^{f_{tm}}\dots)((\overline{h+1})^{f_{\overline{h+1},m+1}}\dots \bar{j}^{f_{\bar{j},m+1}}\dots)$$
$$= a_{\bar{j}}^{(m+1)}(1^{f_{1m}}\dots t^{f_{tm}}\dots)(\bar{j})((\overline{h+1})^{f_{\overline{h+1},m+1}}\dots \bar{j}^{f_{\bar{j},m+1}-1}\dots),$$
$$|[\nu](m')\rangle = (1^{f_{1m}}\dots t^{f_{tm}+1}\dots)((\overline{h+1})^{f_{\overline{h+1},m+1}}\dots \bar{j}^{f_{\bar{j}m+1}-1}\dots)$$
$$= b_t^{\prime(m)}(1^{f_{1m}}\dots t^{f_{tm}}\dots)(t)((\overline{h+1})^{f_{\overline{h+1},m+1}}\dots \bar{j}^{f_{\bar{j}m+1}-1}\dots),$$

where we ignored all irrelevant parentheses, and

$$b_t^{\prime(m)} = b_t^{(m)}(m_{tm} \to m_{tm}+1). \tag{4.20}$$

Therefore

$$M_{m,m+1}^{(t,\bar{j})}(\text{F–B}) = [(f_m+1)f_{m+1}]^{1/2} a_{\bar{j}}^{(m+1)} b_t^{\prime(m)}. \tag{4.19b}$$

Using (3.30*a*) and (3.21*b*) we obtain

$$M_{m,m+1}^{(t,\bar{j})}(\text{F–B}) = (-1)^{1/2}(-1)^{\sum_{i+1}^{h+1}(m_{\bar{i}m+1}-m_{\bar{i}m})}$$
$$\times\left(\frac{\prod_{i=1}^{h+1}(m_{\bar{i}m+1}-m_{\bar{j}m}+j-i)\prod_{i=1}^{m}(m_{im-1}-m_{tm}+t-i-1)}{\prod_{1, i\neq j}^{h+1}(m_{\bar{i}m}-m_{\bar{j}m}+j-i)\prod_{1, i\neq t}^{m}(m_{im}-m_{tm}+t-i-1)}\right)^{1/2}. \tag{4.19c}$$

(iii) *Fermion–fermion*

Consider the case where a state κ ($\kappa \geq m+2$) in the topmost row of the jth column of a Weyl tableau $W_{(m)}^{[\nu]}$ is being changed into $\kappa-1$ under $E^{\kappa-1}{}_{\kappa}$. Similarly we have the matrix element

$$M_{\kappa-1,\kappa}^{(\bar{j})}(\text{F–F}) = \left\langle \begin{pmatrix} m_{\bar{1}m+n}\dots m_{\overline{h+n},m+n} \\ \dots \\ m_{\bar{1}\kappa}\dots m_{\bar{j}\kappa}\dots m_{\overline{h+k},\kappa} \\ m_{\bar{1}\kappa-1}\dots m_{\bar{j}\kappa-1}+1\dots \\ \dots \\ m_{11} \end{pmatrix} \middle| E^{\kappa-1}{}_{\kappa} \middle| \begin{pmatrix} m_{\bar{1}m+n}\dots m_{\overline{h+n},m+n} \\ \dots \\ m_{\bar{1}\kappa}\dots m_{\bar{j}\kappa}\dots m_{\overline{h+k},\kappa} \\ m_{\bar{1}\kappa-1}\dots m_{\bar{j}\kappa-1}\dots \\ \dots \\ m_{11} \end{pmatrix} \right\rangle$$
$$= [(f_{\kappa-1}+1)f_\kappa]^{1/2} a_{\bar{j}}^{(\kappa)} b_{\bar{j}}^{\prime(\kappa-1)}, \tag{4.21a}$$

where

$$b_{\bar{j}}^{\prime(\kappa-1)} = b_{\bar{j}}^{(\kappa-1)}(m_{\bar{j}\kappa-1} \to m_{\bar{j}\kappa-1}+1). \tag{4.22}$$

Using (3.30), we get

$$M_{\kappa-1,\kappa}^{(\bar{j})}(\text{F–F}) = (-1)^{\sum_{j+1}^{h+k}(m_{\bar{i}\kappa}-m_{\bar{i}\kappa-1})}(-1)^{\sum_{1}^{j-1}(m_{\bar{i}\kappa-1}-m_{\bar{i}\kappa-2})}$$
$$\times(-1)^{1/2}\left(\frac{\prod_{i=1}^{h+k}(m_{\bar{i}\kappa}-m_{\bar{j}\kappa-1}+j-i)\prod_{i=1}^{h+k-1}(m_{\bar{i}\kappa-2}-m_{\bar{j}\kappa-1}+j-i-1)}{\prod_{1, i\neq j}^{h+k}(m_{\bar{i}\kappa-1}-m_{\bar{j}\kappa-1}+j-i)\prod_{1, i\neq j}^{h+k-1}(m_{\bar{i}\kappa-1}-m_{\bar{j}\kappa-1}+j-i-1)}\right)^{1/2}, \tag{4.21b}$$

where $\kappa = m+k$, $h = m_{1m}$.

The matrix elements of E'_{i-1} can be obtained from those of $E^{i-1}{}_i$ through

$$\left\langle\begin{pmatrix}[\nu]\\(m')\end{pmatrix}\middle|E^{i-1}{}_i\middle|\begin{pmatrix}[\nu]\\(m)\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}[\nu]\\(m)\end{pmatrix}\middle|E^{i}{}_{i-1}\middle|\begin{pmatrix}[\nu]\\(m')\end{pmatrix}\right\rangle. \tag{4.23}$$

Thus (4.16), (4.19) and (4.21) exhausted all possible cases for $E^i{}_j$ with $i=j\pm1$.

Furthermore, from the commutator (Dondi and Jarvis 1981)

$$(E^A{}_B, E^C{}_D)_{-\left[{AC \atop BD}\right]}=\delta_{BC}E^A{}_D-\left[{AC \atop BD}\right]\delta_{AD}E^C{}_B$$

and the matrices of $E^i{}_j$ with $i=j\pm1$, we can get the matrix for any generator $E^A{}_B$ of $U(m/n)$.

5. The norm $R^{[\nu]2}$

Finally, let us find a closed form for the norm $R^{[\nu]s}(\{f_i\})$ in (2.8*b*). Forming the scalar product of $|{[\nu] \atop r,(m)}\rangle$ with equation (2.8*b*), using (2.27), (2.21*b*) and (2.23), we can easily obtain

$$R^{[\nu]s}(\{f_i\})=\mathcal{N}^{-1/2}\langle[\nu](m)|[\nu]s\rangle. \tag{5.1}$$

It shows that the norm is related to the overlap between the non-standard basis $|[\nu](m)\rangle$ and the Yamanouchi basis $|[\nu]s\rangle$ of $S(f)$. Usually there are several projection operators $P_r^{[\nu]s}$ with different s which will lead to the same Gel'fand basis vector $|{[\nu] \atop r,(m)}\rangle$ upon acting on $|\Phi_0\rangle$. Let s_0 be the minimum possible Yamanouchi number. The Yamanouchi symbol s_0 is easily obtainable from the extended Yamanouchi symbol for $|[\nu](m)\rangle$ in (3.10*a*) by inserting the inverted parentheses ')(' between any two different row numbers as well as between any two different column numbers. Therefore if

$$\begin{aligned}|[\nu](m)\rangle=&(1^{f_{11}})(1^{f_{12}}2^{f_{22}})\dots(1^{f_{1k}}\dots j^{f_{jk}}\dots k^{f_{kk}})\dots\\&\times((\overline{h+n})^{f_{\overline{h+n},m+n}}\dots\bar{1}^{f_{\bar{1}m+n}})\dots((\overline{h+k})^{f_{\overline{h+k},m+k}}\dots\bar{j}^{f_{\bar{j}m+k}}\dots\bar{1}^{f_{\bar{1}m+k}})\dots\end{aligned} \tag{5.2}$$

then the corresponding $|[\nu]s_0\rangle$ will be

$$\begin{aligned}|[\nu]s_0\rangle=&(1^{f_{11}})(1^{f_{12}})(2^{f_{22}})\dots(1^{f_{1k}})\dots(j^{f_{jk}})\dots(k^{f_{kk}})\dots\\&\times(\overline{h+n})^{f_{\overline{h+n},m+n}}\dots(\bar{1}^{f_{\bar{1}m+n}})\dots(\overline{h+k})^{f_{\overline{h+k},m+k}}\dots(\bar{j}^{f_{\bar{j}m+k}})\dots(\bar{1}^{f_{\bar{1}m+k}})\dots\end{aligned} \tag{5.3}$$

The overlap $\langle[\nu](m)|[\nu]s_0\rangle$ can be factorised into two parts

$$\langle[\nu](m)|[\nu]s_0\rangle=\langle[\nu](m)|[\nu]s_0\rangle_B\langle[\nu](m)|[\nu]s_0\rangle_F, \tag{5.4}$$

and $\langle[\nu](m)|[\nu]s_0\rangle_B(\langle[\nu](m)|[\nu]s_0\rangle_F)$ can be calculated by successively decomposing all the symmetric (antisymmetric) parentheses in $|[\nu](m)\rangle$. One may either use the left or right decomposition. For definiteness, the left decomposition is used here.

For multiple decomposition, we need to introduce the following notation for the decomposition coefficients. Suppose the kth parenthesis in (5.2) has already been decomposed to

$$(B)=(1^{f_{1k}})(2^{f_{2k}})\dots(j-1)^{f_{j-1,k}}(j)^l(j^{f_{jk}-l}(j+1)^{f_{j+1,k}}\dots k^{f_{kk}}). \tag{5.5}$$

The next decomposition is

$$(\mathrm{B}) = a((1^{f_{1k}})\dots(j^{l+1}))\cdot(1^{f_{1k}})(2^{f_{2k}})\dots(j-1)^{f_{j-1,k}}(j)^{l+1}(j^{f_{jk}-l-1}(j+1)^{f_{j+1,k}}\dots k^{f_{kk}}), \tag{5.6}$$

where $a((1^{f_{1k}})\dots(j^{l+1}))$ is the decomposition coefficient which depends on the 'history' of the decomposition. According to the independent decomposition rule in § 3, we know that whenever a row number i has been separated from the kth symmetric parenthesis leftwards, the integer m_{ik-1} should be increased by one in using formula (3.21a) for further decomposition, i.e. let $m_{ik-1}\to m_{ik-1}+1$. Hence we have

$$a((1^{f_{1k}})\dots(j^{l+1})) = a_j^{(k)}(m_{ik-1}\to m_{ik-1}+f_{ik}, i=1,\dots j-1; m_{jk-1}\to m_{jk-1}+l). \tag{5.7}$$

Noting that $m_{ik-1}+f_{ik}=m_{ik}$ and using (3.21a) we found

$$a((1^{f_{1k}})\dots(j^{l+1})) = (f_k - f_{1k} - \dots - f_{j-1k} - l)^{-1/2}\left(\frac{\prod_{i=j}^{k}(m_{ik}-m_{jk-1}+j-i-l)}{\prod_{i=j+1}^{k}(m_{ik-1}-m_{jk-1}+j-i-l)}\right)^{1/2}. \tag{5.8}$$

Define

$$\langle[\nu](m)\|[\nu]s_0\rangle'_{\mathrm{B}} = (f_1!\dots f_m!)^{1/2}\langle[\nu](m)\|[\nu]s_0\rangle_{\mathrm{B}}. \tag{5.9a}$$

Comparing (5.2) with (5.3) and using (5.8) we get

$$\langle[\nu](m)\|[\nu]s_0\rangle'_{\mathrm{B}} = \prod_{k=1}^{m}(f_{M_k k}!)^{1/2}\prod_{j=1}^{M_k-1}{}'\prod_{l=0}^{f_{jk}-1}\left(\frac{\prod_{i=j}^{k}(m_{ik}-m_{jk-1}+j-i-l)}{\prod_{i=j+1}^{k}(m_{ik-1}-m_{jk-1}+j-i-l)}\right)^{1/2}, \tag{5.9b}$$

where M_k is the maximum row number in the kth parenthesis, while the prime restricts the range of the running index j to that for which $f_{jk}\geqslant 1$.

In analogy with (5.6), for the κ $(=m+k)$th antisymmetric parenthesis we have

$$\begin{aligned}(\overline{h+k})^{f_{\overline{h+k},\kappa}}&\dots(\overline{j+1})^{f_{\overline{j+1},\kappa}}(\bar{j})^{l}(\bar{j}^{f_{\bar{j}\kappa}-1}\dots \bar{1}^{f_{\bar{1}\kappa}})\\ &= a((\overline{h+k})^{f_{\overline{h+k},\kappa}}\dots(\bar{j})^{l+1})\\ &\quad\times(\overline{h+k})^{f_{\overline{h+k},\kappa}}\dots(\overline{j+1})^{f_{\overline{j+1},\kappa}}(\bar{j})^{l+1}(\bar{j}^{f_{\bar{j}\kappa}-l-1}\dots\bar{1}^{f_{\bar{1}\kappa}}).\end{aligned} \tag{5.10}$$

Analogously, whenever a column number i has been separated from the κth antisymmetric parenthesis leftwards, the substitution $m_{\bar{i}\kappa-1}\to m_{\bar{i}\kappa-1}+1$ has to be made in (3.30) for further decomposition. Thus

$$\begin{aligned}a((\overline{h+k})^{f_{\overline{h+k},\kappa}}&\dots(\bar{j})^{l+1})\\ &= a_{\bar{j}}^{(\kappa)}(m_{\bar{i}\kappa-1}\to m_{\bar{i}\kappa-1}+f_{\bar{i}\kappa}, i=j+1\dots h+k; m_{\bar{j}\kappa-1}\to m_{\bar{j}\kappa-1}+l).\end{aligned} \tag{5.11}$$

Noting that $m_{\bar{i}\kappa-1}+f_{\bar{i}\kappa}=m_{\bar{i}\kappa}$ and using (5.30a), we get

$$\begin{aligned}a((\overline{h+k})^{f_{\overline{h+k},\kappa}}&\dots(\bar{j})^{l+1})\\ &= (f_\kappa - f_{\overline{h+k},\kappa} - \dots - f_{\overline{j+1},\kappa} - l)^{-1/2}\left(\frac{\prod_{i=1}^{j}(m_{\bar{i}\kappa}-m_{\bar{j}\kappa-1}+j-i-l)}{\prod_{i=1}^{j-1}(m_{\bar{i}\kappa-1}-m_{\bar{j}\kappa-1}+j-i-l)}\right)^{1/2}.\end{aligned} \tag{5.12}$$

Define

$$\langle[\nu](m)|[\nu]s_0\rangle'_{\mathrm{F}} = (f_{m+1}!\ldots f_{m+n}!)^{1/2}\langle[\nu](m)|[\nu]s_0\rangle_{\mathrm{F}}. \tag{5.13a}$$

Comparing (5.2) with (5.3) and using (5.12), we obtain

$$\langle[\nu](m)|[\nu]s_0\rangle'_{\mathrm{F}} = \prod_{\kappa=m+1}^{m+n} (f_{\bar{M}_\kappa\kappa}!)^{1/2} \prod_{j=\bar{M}_\kappa-1}^{h-m+\kappa}{}' \prod_{l=0}^{f_{\bar{j}\kappa}-1} \left(\frac{\prod_{i=1}^{j}(m_{\bar{i}\kappa} - m_{\bar{j}\kappa-1} + j - i - l)}{\prod_{i=1}^{j-1}(m_{\bar{i}\kappa-1} - m_{\bar{j}\kappa-1} + j - i - l)} \right)^{1/2}. \tag{5.13b}$$

where $\bar{M}_\kappa$ denotes the minimum column number in the κth antisymmetric parenthesis, and the running index j is restricted to those for which $f_{\bar{j}\kappa} > 1$.

Combining (2.24), (5.1), (5.4), (5.9) and (5.13), we get the sought for norm

$$R^{[\nu]s_0}(\{f_i\}) = \langle[\nu](m)|[\nu]s_0\rangle'_{\mathrm{B}}\langle[\nu](m)|[\nu]s_0\rangle'_{\mathrm{F}}. \tag{5.15}$$

From (5.9*b*) and (5.13*b*) it is seen that the norm $R^{[\nu]s_0}$ is always real positive, in conformity with our previous phase convention (Chen *et al* 1983b).

The norm $R^{[\nu]s}$ for $s \neq s_0$ can be found from $R^{[\nu]s_0}$ by using equation (42*b*) in Chen *et al* (1983b).

Before concluding the present paper, let us mention an important fact that by specialising the general non-standard basis $|[\nu](m)\rangle$ of (2.19) to the simplest case of the $S(f) \supset S(f_1) \otimes S(f_2)$ IRB, the overlap $\langle[\nu](m)|[\nu]s_0\rangle_{\mathrm{B}}$ $(\langle[\nu](m)|[\nu]s_0\rangle_{\mathrm{F}})$ become the transformation coefficients between the Yamanouchi basis and the $S(f) \supset S(f_1) \otimes S(f_2)$ IRB with irreps of $S(f_2)$ being totally symmetric (antisymmetric). These transformation coefficients of the permutation group are calculable from (5.9) and (5.13). Equation (5.9) and (5.13) have been verified from the agreement between the thus calculated transformation coefficients with the numerical table obtained via a FORTRAN program based on the eigenfunction method (Chen *et al* 1983c).

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